Symmetrization of the self-energy integral in the Yakhot-Orszag renormalization-group calculation

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A further modification (through proper symmetrization of the self-energy integral) on Wang and Wu's modified calculations [Phys. Rev. E **48**, 37 (1993)] reproduces Yakhot and Orszag's result [J. Sci. Comput. **1**, 3 (1986)]. [S1063-651X(97)01602-4]

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The dynamic renormalization-group (RG) approach of Ma and Mazenko [1] has been mainly a tool to study the large-scale long-time properties of Navier-Stokes fluids driven by a random external noise, first used by Forster, Nelson, and Stephen [2]. A generalization given by DeDominicis and Martin [3] includes the Kolmogorov spectrum of strong turbulence for a particular value of a parameter $\epsilon(=4)$ coming from the correlation of the random external stirring. Yakhot and Orszag [4,5] carried out the renormalization-group calculation based on these ideas, and obtained various universal amplitudes associated with Kolmogorov turbulence (including the case of a passive scalar), in remarkable agreement with experimental numbers. However, Yakhot and Orszag used a ϵ-expansion scheme (commonly used in critical phenomena) in their calculations, where one sets $\epsilon=0$ in the calculated amplitudes (which is equivalent to extracting the ultraviolet pole in the self-energy integral). This has led to objections [6], following which Wang and Wu [7] have suggested a modification of the RG calculation.

In this paper, we would like to show that a further modification in Wang and Wu's calculations, through proper symmetrization of the self-energy integral, gives back the result of the Yakhot and Orszag calculation.

The inertial range turbulence has been modeled by the randomly driven Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\frac{\nabla P}{\rho_0}+\nu_0\nabla^2\mathbf{u}+\mathbf{f} \tag{1}$$

along with the incompressibility condition

$$\nabla\cdot\mathbf{u}=0, \tag{2}$$

where $\mathbf{u}(\mathbf{x},t)$ and $P(\mathbf{x},t)$ are the velocity and pressure fields, ρ_0 the density, and ν_0 the kinematic viscosity of the fluid; the dynamics, having been modeled to be driven by the random stirring force $\mathbf{f}(\mathbf{x},t)$, have been assumed to have a Gaussian white-noise statistics with the correlation

$$\langle f_i(\mathbf{k},\omega)f_j(\mathbf{k}',\omega')\rangle=F(k)P_{ij}(\mathbf{k})[2\pi]^d\delta^d(\mathbf{k}+\mathbf{k}')\times[2\pi]\delta(\omega+\omega') \tag{3}$$

in the Fourier space, where $P_{ij}(\mathbf{k})=(\delta_{ij}-k_ik_j/|\mathbf{k}|^2)$, and

$$F(k)=\frac{2D_0}{k^y}. \tag{4}$$

In the Fourier-transformed space, Eqs. (1) and (2) take the form

$$(-i\omega+\nu_0k^2)u_i(\mathbf{k},\omega)=f_i(\mathbf{k},\omega)-\frac{i\lambda_0}{2}P_{ijl}(\mathbf{k})\int\frac{d^d\mathbf{q}d\omega'}{[2\pi]^{d+1}}\int\frac{d^d\mathbf{p}d\omega''}{[2\pi]^{d+1}}\times u_j(\mathbf{q},\omega')u_l(\mathbf{p},\omega'')[2\pi]^d\delta^d(\mathbf{q}+\mathbf{p}-\mathbf{k})\times[2\pi]\delta(\omega'+\omega''-\omega) \tag{5}$$

and

$$k_ju_j(\mathbf{k},\omega)=0, \tag{6}$$

where $P_{ijl}(\mathbf{k})=k_jP_{il}(\mathbf{k})+k_lP_{ij}(\mathbf{k})$ and λ_0 $(=1)$ is the formal expansion parameter. An ultraviolet cutoff at a wave number Λ to the wave-vector integration is assumed, corresponding to the ''internal'' (viscous) cutoff.

Now, one eliminates (i.e., integrates away) the ''fast'' modes $\mathbf{u}^>(\mathbf{k},\omega)$ lying in the band $\Lambda e^{-r}<k<\Lambda$, leading to an equation for the ''slow'' modes $\mathbf{u}^<(\mathbf{k},\omega)$ (belonging to $0<k<\Lambda e^{-r}$) given by

$$(-i\omega+\nu_0k^2)u_i^<(\mathbf{k},\omega)=f_i^<(\mathbf{k},\omega)-\frac{i\lambda_0}{2}P_{ijl}(\mathbf{k})\int\frac{d^d\mathbf{q}d\omega'}{[2\pi]^{d+1}}\int\frac{d^d\mathbf{p}d\omega''}{[2\pi]^{d+1}}\times u_j^<(\mathbf{q},\omega')u_l^<(\mathbf{p},\omega'')[2\pi]^d\delta^d(\mathbf{q}+\mathbf{p}-\mathbf{k})\times[2\pi]\delta(\omega'+\omega''-\omega)+R_i(\mathbf{k},\omega), \tag{7}$$

with

$$R_i(\mathbf{k},\omega)=-\Sigma_{ij}(\mathbf{k},\omega)u_j^<(\mathbf{k},\omega), \tag{8}$$

which, when taken on to the left-hand side in Eq. (7), gives correction to the bare ''viscosity'' $\nu_0 k^2$, given by the self-energy

$$-\Sigma_{ik}(\mathbf{k},\omega) = 4\left(-\frac{i\lambda_0}{2}\right)P_{ijl}(\mathbf{k})\int\frac{d^d\mathbf{q}\,d\omega'}{[2\pi]^{d+1}}\int\frac{d^d\mathbf{p}\,d\omega''}{[2\pi]^{d+1}} \times Q^{>}_{jm}(\mathbf{q},\omega')G^{>}_{ln}(\mathbf{p},\omega'')\left(-\frac{i\lambda_0}{2}\right)P_{nmk}(\mathbf{p}) \times[2\pi]^d\delta^d(\mathbf{q}+\mathbf{p}-\mathbf{k})[2\pi]\delta(\omega'+\omega''-\omega) \qquad (9)$$

with $G_{ij}(\mathbf{k},\omega)=(-i\omega+\nu_0k^2)^{-1}P_{ij}(\mathbf{k})$ being the propagator and $Q_{ik}=G_{ij}F_{jl}G^*_{lk}$ the velocity correlation. Using the property of isotropy, $X_{ij}(\mathbf{k})=X(k)P_{ij}(\mathbf{k})$, and carrying out the frequency integrations, we obtain from Eq. (9)

$$\Sigma_{ik}(\mathbf{k},\omega)=\lambda_0^2P_{ijl}(\mathbf{k})\int\frac{d^d\mathbf{q}}{[2\pi]^d}\int\frac{d^d\mathbf{p}}{[2\pi]^d}P_{jm}(\mathbf{q})P_{lmk}(\mathbf{p}) \times\frac{F(q)}{2\nu_0q^2}\frac{1}{-i\omega+\nu_0q^2+\nu_0p^2} \times[2\pi]^d\delta^d(\mathbf{q}+\mathbf{p}-\mathbf{k}). \qquad (10)$$

Now, using the δ function in Eq. (10) to integrate *only* over $\mathbf{p}$ leads to

$$\Sigma_{ik}(\mathbf{k},\omega)=\lambda_0^2P_{ijl}(\mathbf{k})\int\frac{d^d\mathbf{q}}{[2\pi]^d}P_{jm}(\mathbf{q})P_{lmk}(\mathbf{k}-\mathbf{q}) \times\frac{F(q)}{2\nu_0q^2}\frac{1}{-i\omega+\nu_0q^2+\nu_0|\mathbf{k}-\mathbf{q}|^2}. \qquad (11)$$

At this point, Yakhot and Orszag make the substitution $\mathbf{q}\to\mathbf{q}-\mathbf{k}/2$ in Eq. (11), and evaluate the integral in the limit $k\to0$ and $\omega\to0$ by extracting the leading contribution from the region $q\gg k$, yielding

$$\Sigma^{\rm YO}_{ij}(\mathbf{k},\omega)=\left[\frac{S_d}{[2\pi]^d}\frac{d^2-d-\epsilon}{2d(d+2)}\right]\frac{\lambda_0^2D_0}{\nu_0^2}\left(\frac{\epsilon^{\epsilon r}-1}{\epsilon\Lambda^\epsilon}\right)k^2P_{ij}(\mathbf{k}) \qquad (12)$$

where

$$\epsilon=4+y-d \qquad (13)$$

is the small parameter of the RG theory, which is to be set to zero in the square bracket.

Wang and Wu do not make the substitution $\mathbf{q}\to\mathbf{q}-\mathbf{k}/2$ in Eq. (11). Using expansions similar to those of Yakhot and Orszag, their calculations yield

$$\Sigma^{\rm WW}_{ij}(\mathbf{k},\omega)=\left[\frac{S_d}{[2\pi]^d}\frac{d^2-d}{2d(d+2)}\right]\frac{\lambda_0^2D_0}{\nu_0^2}\left(\frac{e^{\epsilon r}-1}{\epsilon\Lambda^\epsilon}\right)k^2P_{ij}(\mathbf{k}) \qquad (14)$$

where the quantity in square brackets is found to be independent of ϵ, alleviating one from setting $\epsilon=0$.

However, we point out that there is no reason to prefer to do the $\mathbf{p}$ integration first in Eq. (10). It is also equally possible to do the $\mathbf{q}$ integration first. Adding the results of the two integrations gives

$$2\Sigma_{ik}(\mathbf{k},\omega)=\lambda_0^2P_{ijl}(\mathbf{k})\left[\int\frac{d^d\mathbf{q}}{[2\pi]^d}P_{jm}(\mathbf{q}) \times P_{lmk}(\mathbf{k}-\mathbf{q})\frac{F(q)}{2\nu_0q^2}\frac{1}{-i\omega+\nu_0q^2+\nu_0|\mathbf{k}-\mathbf{q}|^2} + \int\frac{d^d\mathbf{p}}{[2\pi]^d}P_{jm}(\mathbf{k}-\mathbf{p})P_{lmk}(\mathbf{p}) \times\frac{F(|\mathbf{k}-\mathbf{p}|)}{2\nu_0|\mathbf{k}-\mathbf{p}|^2}\frac{1}{-i\omega+\nu_0|\mathbf{k}-\mathbf{p}|^2+\nu_0p^2}\right]. \qquad (15)$$

Evaluating the integrals in the RG limit $k\to0$ and $\omega\to0$ by picking up the leading contribution from the regions $q\gg k$ and $p\gg k$, we find from Eq. (15)

$$\Sigma_{ij}(\mathbf{k},\omega)=\left[\frac{S_d}{[2\pi]^d}\frac{1}{2}\left\{\frac{d^2-d}{2d(d+2)}+\frac{d^2+d-8-2y}{2d(d+2)}\right\}\right] \times\frac{\lambda_0^2D_0}{\nu_0^2}\left(\frac{e^{\epsilon r}-1}{\epsilon\Lambda^\epsilon}\right)k^2P_{ij}(\mathbf{k}), \qquad (16)$$

the first term being the (half of) Wang and Wu's result, whereas, the second term does depend on y (and therefore on ϵ).

Using Eq. (13), it can easily be seen that this result [Eq. (16)] reduces to the Yakhot-Orszag result, Eq. (12). It should be noted that, quite like Wang and Wu's calculations, we did not make any replacement like $\mathbf{q}\to\mathbf{q}-\mathbf{k}/2$ or $\mathbf{p}\to\mathbf{p}-\mathbf{k}/2$ to get the result [Eq. (16)] from Eq. (15).

We have thus made a proper symmetrization of the self-energy integral [given by Eq. (10)] by giving no preference to one integral over the other through the use of the δ function. Yakhot and Orszag's RG calculations achieve the symmetrization through the substitution $\mathbf{q}\to\mathbf{q}-\mathbf{k}/2$ in Eq. (11). Although it changes the limit of integration to $\Lambda e^{-r}<|\mathbf{q}-\mathbf{k}/2|<\Lambda$, this substitution reproduces the result in Eq. (16) in the leading order mainly because $k\ll\Lambda$.

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